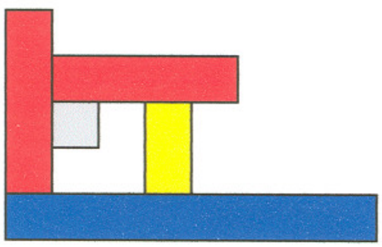
Infinifield Math Contest

In the game Infinifield, every legal game board (whether 2-dimensional or 3-dimensional, regardless of complexity) can be transformed into a 2-dimensional, non-overlapping network. This makes it much easier for one to locate the "dead-ends" and the "safe regions" of the playing field. Of course, since pieces are removed from the game board as the game progresses, "safe regions" may later become "dead-ends".

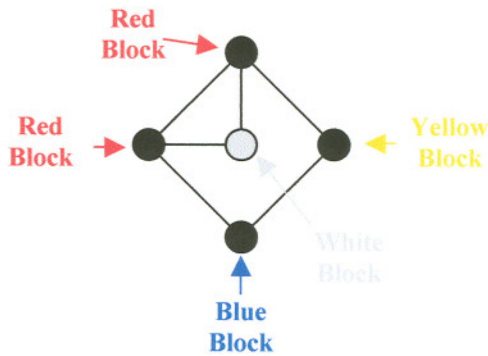
Transforming the game board into a network also makes it easier for one to approach the math problem in your contest. By analyzing the networks of game boards, I was able to determine that there are $\dfrac{\sum_{i=1}^{7}\binom{i+7}{i}}{2}+15 = 3,232$ *distinct* playing fields that can be legally built in the shape of a 1×64 rectangle.

Before I derive this formula, I will begin by demonstrating how to transform a game board into a network. In the network, each block is represented by a dot (node). If two blocks touch each other, the corresponding dots (nodes) are connected with a line segment (a branch). Thus, in the example below, the game board on the left would transform into the network on the right.

Game Board

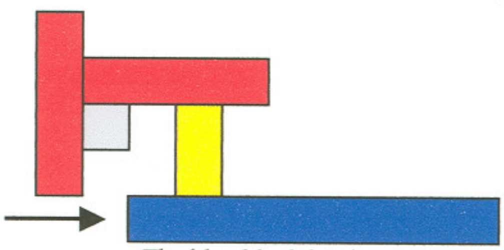


Corresponding Network



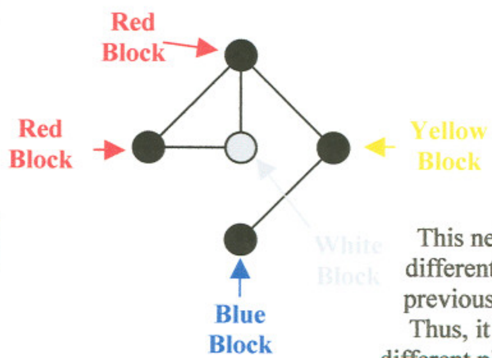
Suppose that you modify the game board above by sliding the blue block to the right so that it no longer touches the red block on its left. You can see from the corresponding network that this would create a different playing field.

Game Board



The blue block has been moved to the right so that it no longer touches the red block.

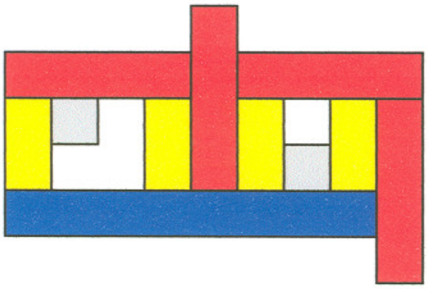
Corresponding Network



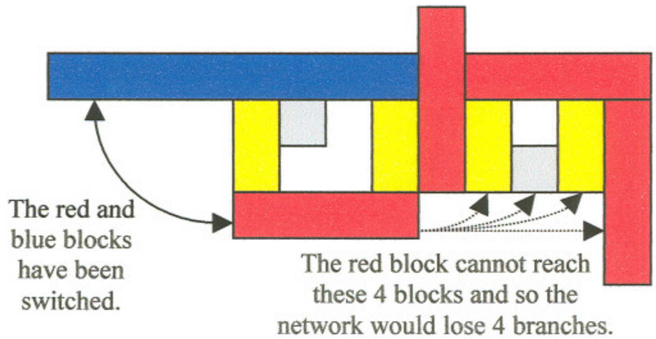
This network is different from the previous network. Thus, it creates a different playing field.

Switching blocks is another modification that you can make to a game board to create new playing fields. Whenever a long, blue block touches so many different blocks that it would be impossible for a shorter, red block to reach them all, switching the two blocks creates a different network and thus a different playing field (see example below).

Game Board A



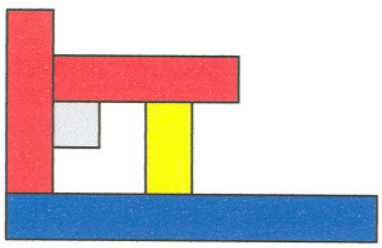
Game Board B



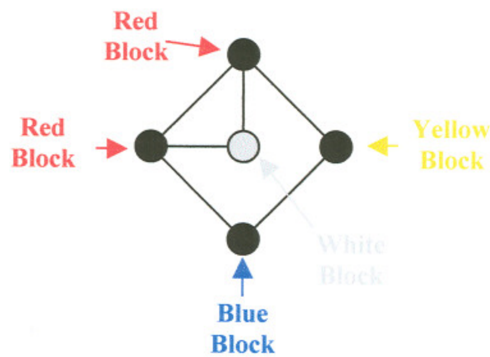
However, there are instances where switching blocks in a game board would *not* result in the creation of a new playing field. Obviously, if we switched a red block with another red block, the playing field would stay exactly the same.

It is also the case that you can sometimes switch blocks that are different colors without changing the corresponding network. Below is an example of a game board in which one could switch a red block with a *blue* block and the new game board would still correspond to the same network. In cases like these, switching different colored blocks does *not* result in the creation of a new *distinct* playing field.

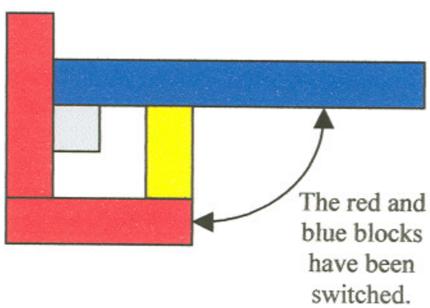
Game Board A



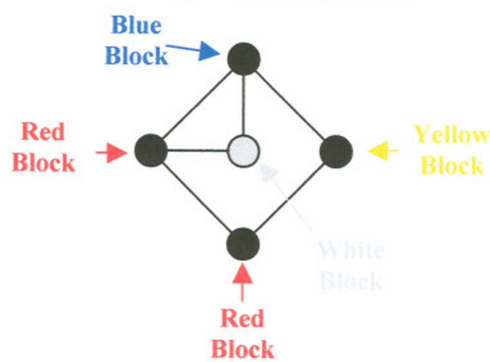
Corresponding Network



Game Board B



Corresponding Network



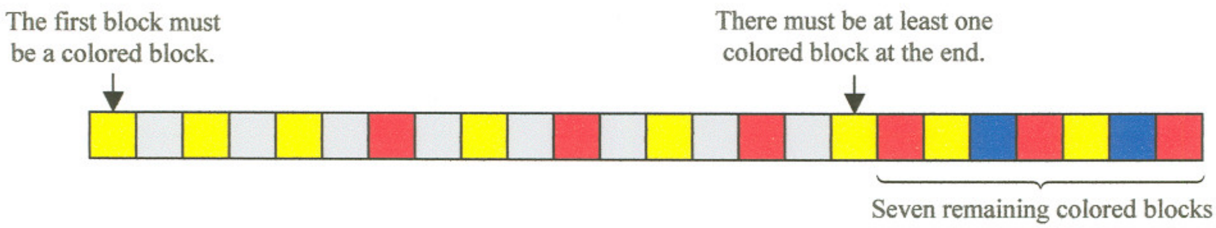
This is the same network as above.
Thus, the playing fields are identical.

Recognizing that switching colored blocks might not result in a different playing field is very important to the solution of this math problem. One might be tempted to use a formula that counts the permutations of different colored blocks in order to calculate the number of playing fields. However, this would result in a large amount of over-counting because each time you counted a *distinct* playing field, you would also be counting the $\frac{16!}{8!\cdot 6!\cdot 2!} = 360,360$ permutations of the different colored blocks.

Now that I have demonstrated how to transform a game board into a network and how the network determines if two playing fields are distinct, I can derive the formula for how to calculate the number of *distinct* playing fields that can be legally built in the shape of a 1×64 rectangle.

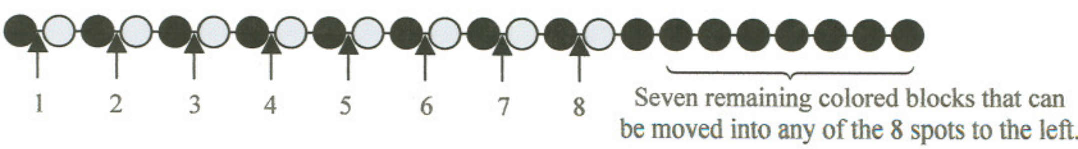
According to the rules of the game, each white block (gray in my picture) must touch at least two colored blocks and may *not* touch another white block. So, a good first model for a 1×64 game board would be the one shown below.

The first block must be a colored block because a white block must touch at least two blocks. White blocks are not allowed to touch each other, so after the first block, alternate a white block with a colored block. A white block cannot be the last block since it must touch at least two blocks, so follow the last white block with a colored block. There will be seven remaining colored blocks, so attach them to the end of the rectangle.



Each block counts as one square on the game board regardless of how long the block is. Since the blocks are laid end-to-end, the corresponding network is always going to be a straight line with 24 nodes (in the network below, the gray dots represent white blocks and the black dots represent colored blocks). Notice that you could switch *any* colored block with another colored block and it would *not* create a different network.

However, the number of colored blocks at each end of the game board and the number of colored blocks between each pair of white blocks *is* relevant to the calculation of distinct playing fields. This is due to the fact that the white blocks are the starting points for the pawn war. Thus, between 1 and 7 of the remaining blocks at the right can be repositioned into any of the 8 spots shown below to create a new, *distinct* playing field (the first spot is to the left of the first white block and there are seven more spots that are between pairs of white blocks – see below).



To calculate all of the ways in which we could reposition those 7 remaining colored blocks into the 8 spots, begin by moving only one of the colored blocks into one of the 8 spots. Since the order of the colored blocks doesn't matter, there are only 8 ways to do this. The order of the colored blocks is irrelevant because switching a colored block with another colored block creates a permutation that corresponds to the exact same network. Thus, the permutations do *not* represent different playing fields.

Next, move exactly two of those 7 colored blocks into one or two of the 8 spots. We could insert both blocks into the same spot, or we could insert one block into one of the spots and the other block into a different spot.

Continue in this fashion until you have moved all 7 blocks into the 8 spots.

There is theorem in combinatorics that would be helpful with this calculation. It that states that the number of ways to place r objects into n spots, where the order doesn't matter, and more than one object can go into the same spot, is $\binom{r+(n-1)}{r}$.

When we move one block, there are $\binom{1+(8-1)}{1}=\binom{1+7}{1}=8$ distinct ways to do this.

When we move two blocks, there are $\binom{2+(8-1)}{2}=\binom{2+7}{2}=36$ distinct ways to do this.

When we move 7 blocks, there are $\binom{7+(8-1)}{7}=\binom{7+7}{7}=3,432$ distinct ways to do this.

The calculation for summing up these seven combinations plus the original game board is $1+\sum_{i=1}^{7}\binom{i+7}{i}$ which is equal to 6,435.

However, there is still a source of over-counting in this calculation. With respect to the line cutting the game board in half, many of the game boards have mirror images that were also counted. For example, if you move all seven blocks to the far left, you get the mirror image of the original game board.

Game boards that are mirror images of each other correspond to the same network and thus do not represent *distinct* playing fields. In this calculation, every playing field was counted twice (we counted each mirror image) except for playing fields whose game boards are symmetrical with respect to the line cutting the game board in half. Those game boards are mirror images of themselves, so those playing fields were counted only once. There are $1+\sum_{i=1}^{3}\left(i\cdot\binom{4}{i}\right)=29$ of these playing fields.

To adjust for this, subtract the 29 playing fields that were only counted once (don't worry, we'll add them back at the end). That leaves 6,406 playing fields that were double-counted. Now, divide by two, and then add back the 29 playing fields.

The calculation can be expressed as:

$$\frac{1+\sum_{i=1}^{7}\binom{i+7}{i}-29}{2}+29 \;=\; \frac{\sum_{i=1}^{7}\binom{i+7}{i}}{2}+15 \;=\; 3,232 \text{ \textit{distinct} playing fields.}$$

$\square$